

THE NASH CONJECTURE FOR NONPROJECTIVE THREEFOLDS

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1. INTRODUCTION

In his fundamental paper on the topology of real algebraic varieties Nash formulated the following bold conjecture:

Conjecture 1.1. [Nash52] *For every compact differentiable manifold M of dimension n there is a real algebraic variety X such that M is diffeomorphic to $X(\mathbb{R})$ and*

1. X is smooth,
2. X is projective, and
3. X is birational to \mathbb{P}^n .

It turns out that this fails for surfaces [Comessatti14] and recently it was proved that this also fails in dimension 3:

Theorem 1.2. [Kollár98c] *Let X be a smooth, real, projective 3-fold birational to \mathbb{P}^3 . Assume that $X(\mathbb{R})$ is orientable. Then every connected component of $X(\mathbb{R})$ is among the following:*

1. Seifert fibered,
2. connected sum of several copies of S^3/\mathbb{Z}_{m_i} (called lens spaces),
3. torus bundle over S^1 or doubly covered by a torus bundle over S^1 ,
4. finitely many other possible exceptions, or
5. obtained from one of the above 3-manifolds by repeatedly taking connected sum with \mathbb{RP}^3 and $S^1 \times S^2$.

Therefore it is of interest to consider weaker versions of the conjecture. [Nash52] and [Tognoli73] proved that the conjecture holds if we drop the condition (1.1.3). In dimension 3, [Benedetti-Marin92] prove that the conjecture also holds if we drop the condition (1.1.1) instead. The related “topological Nash conjecture” is solved in dimension 3 by [Akbulut-King91, Benedetti-Marin92] and in general by [Mikhalkin97].

The aim of this paper is to consider the Nash conjecture without the projectivity assumption (1.1.2). It is not completely clear how this could be done. Allowing quasi projective varieties instead of projective ones does not help at all. If Y is quasi projective and $Y(\mathbb{R})$ is compact then there is a projective variety $\bar{Y} \supset Y$ such that $\bar{Y}(\mathbb{R}) = Y(\mathbb{R})$.

To get the right concept, we have to look at compact complex manifolds which can be obtained from \mathbb{P}^n by a sequence of smooth blow ups and blow downs. These manifolds are quite close to projective varieties, thus it was quite a surprise to me that the Nash conjecture holds for them:

Theorem 1.3. *For every compact, connected, differentiable 3-manifold M there is a compact complex manifold X which can be obtained from \mathbb{P}^3 by a sequence of smooth, real blow ups and downs such that M is diffeomorphic to $X(\mathbb{R})$.*

To be precise, a real structure on a complex manifold is a pair (Y, τ) where Y is a complex manifold and $\tau : Y \rightarrow Y$ an antiholomorphic involution. $Y(\mathbb{R})$ denotes the fixed point set of τ . The main problem of the theory of these objects is that all reasonable names have already been taken. “Real analytic space” is used for something else (cf. [Narasimhan68]) and “real complex manifold” sounds goofy.

Complex manifolds which are bimeromorphic to a projective one have been investigated by several authors:

Definition 1.4. [Artin68, Moishezon67] A compact complex manifold Y is called a *Moishezon manifold* or an *Artin algebraic space* if it is bimeromorphic to a projective variety.

By a result of Chow and Kodaira, every smooth Moishezon surface is projective (cf. [BPV84, IV.5]). The first nonprojective examples in dimension 3 were found by Hironaka (see [Hartshorne77, App.B.3]).

Definition 1.5. A *real Moishezon manifold* or a *real algebraic space* is an algebraic space Y with an antiholomorphic involution $\tau : Y \rightarrow Y$.

It is not hard to see that if (Y, τ) is a real algebraic space then there is a projective real algebraic variety (Y', τ') and a conjugation invariant bimeromorphic map $\phi : Y \dashrightarrow Y'$.

The proof of (1.3) follows the pattern established in [Benedetti-Marin92].

It is known that every 3-manifold can be obtained from S^3 by repeated surgery along knots. Thus we are done if we can realize every surgery by a suitable birational transformation of real algebraic spaces. [Benedetti-Marin92] realized that this is possible only for a restricted class of surgeries. They called such surgeries *déchirures* (2.3). It turns out that it is better to restrict to an even smaller class of surgeries. I call these *topological flops*. A surgery along a knot is a topological flop iff its surgery coefficient (2.2) is congruent to $1/2$ modulo 1 (3.5).

[Benedetti-Marin92] gives a complete topological classification of 3-manifolds up to topological flops. Thus we can approach (1.3) in the following 2 steps.

First prove (1.3) up to flops. This was done in [Benedetti-Marin92] with the exception of one class. The remaining example is constructed in section 6.

Second, in (5.1) we prove that every topological flop can be realized by an algebraic flop. This is actually quite delicate. It is clear that this can not be done without performing other birational transformations first. The key step is to make sure that these extraneous transformations do not change the set of real points. This is connected with some interesting open problems in the theory of Moishezon manifolds. A related process of blowing up to obtain a flop was studied by [Fujiki80].

2. SURGERY ON THREE DIMENSIONAL MANIFOLDS

2.1. [Torus and solid torus]

As a general reference, see [Rolfsen76, Chap.2].

Let D^2 be the unit disc ($|u| \leq 1$) $\subset \mathbb{C}_u$ and S^1 the unit circle ($|v| = 1$) $\subset \mathbb{C}_v$. A three-manifold T diffeomorphic to $D^2 \times S^1$ is called a *solid torus*. The boundary of a solid torus

$$\partial T \sim (|u| = 1) \times (|v| = 1) \sim S^1 \times S^1$$

is a *torus*. Any simple closed curve on the torus $S^1 \times S^1$ is isotopic to one of the form

$$C_{a,b} := \text{im}[(|z| = 1) \rightarrow S^1 \times S^1] \quad \text{given by} \quad z \mapsto (z^a, z^b) \quad \text{for } (a, b) = 1.$$

The *meridian* of a solid torus is any curve isotopic to $C_{\pm 1, 0}$. Note that these curves are generators of $\ker[\pi_1(\partial T) \rightarrow \pi_1(T)]$, so their isotopy class is well defined. The *longitude* of a solid torus is any curve isotopic to $C_{0, \pm 1}$. The longitude depends on the choice of a diffeomorphism between the solid torus and $D^2 \times S^1$.

The correspondence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [(u, v) \mapsto (u^a v^b, u^c v^d)]$$

gives an isomorphism

$$GL(2, \mathbb{Z}) \cong \frac{(\text{diffeomorphisms of the torus})}{(\text{modulo isotopy})}.$$

Up to isotopy, the diffeomorphisms of a solid torus are given by

$$(u, z) \mapsto (uz^m, z^{\pm 1}) \quad \text{or} \quad (u, z) \mapsto (\bar{u}z^m, z^{\pm 1}), \quad \text{where } m \in \mathbb{Z}.$$

They correspond to the subgroup

$$\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \subset GL(2, \mathbb{Z}).$$

2.2 (Surgery on 3-manifolds).

Let M be a 3-manifold and $L \subset M$ a *knot*. (That is, an embedded copy of S^1 .) Assume that M is orientable along L . L has a tubular neighborhood which is a solid torus T with boundary $\partial T \sim S^1 \times S^1$. Set $N := M \setminus \text{Int } T$. We can think of M as being glued together from these two pieces $M = T \cup N$. Let $\phi : \partial T \rightarrow \partial T$ be any orientation preserving diffeomorphism. We can glue together T and N using ϕ to obtain another 3-manifold $M_\phi := T \cup_\phi N$. The operation that creates M_ϕ from M is called a *surgery* along L . (If ϕ is orientation reversing, we can compose ϕ with an orientation reversing diffeomorphism of T to see that we do not get anything new.) If we fix a diffeomorphism $h : T \sim D^2 \times S^1$ then a diffeomorphism $\phi : \partial T \rightarrow \partial T$ corresponds to an element of $SL(2, \mathbb{Z})$.

Let $\Psi : T \rightarrow T$ be a diffeomorphism and $\psi := \Psi|_{\partial T}$. It is clear that M is diffeomorphic to M_ψ and, more generally, M_ϕ is diffeomorphic to $M_{\phi \circ \psi}$.

In terms of the matrix of ϕ we see that the surgeries corresponding to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} a & b + ma \\ c & d + mc \end{pmatrix}$$

give diffeomorphic 3-manifolds. From this we conclude that the rational number a/c alone determines M_ϕ . ϕ maps the meridian $C_{1,0} : z \mapsto (z, 1)$ to $C_{a,c} : z \mapsto (z^a, z^c)$. Thus the image of the meridian determines the surgery.

It is important to note that this depends on the choice of the isomorphism $h : T \sim D^2 \times S^1$. A different choice of h changes the matrix of

ϕ to

$$\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - mc & * \\ c & d + mc \end{pmatrix}.$$

Thus $c(\phi) := c$ and $r(\phi) := a/c \pmod{1}$ are independent of the choice of h . $r(\phi) \in \mathbb{Q}/\mathbb{Z}$ is called the *surge coefficient* of ϕ .

In some cases, for instance when L is homologous to zero, there is a more canonical choice for h and one obtains a well defined surgery coefficient $r = a/c \in \mathbb{Q}$. See [Rolfsen76, 9.F] for details.

Definition 2.3. A surgery $M \mapsto M_\phi$ is called *trivial mod 2* if $c(\phi) \equiv 0 \pmod{2}$. This notion is called *déchirure* in [Benedetti-Marin92] where it was first studied in detail.

Lemma 2.4. A surgery $M \mapsto M_\phi$ is trivial mod 2 iff one can choose ϕ such that the induced map

$$\phi_* : H_1(\partial T, \mathbb{Z}_2) \rightarrow H_1(\partial T, \mathbb{Z}_2) \quad \text{is the identity.}$$

In this case there is a natural isomorphism $H_i(M, \mathbb{Z}_2) \rightarrow H_i(M_\phi, \mathbb{Z}_2)$ for every i .

Proof. Choose a basis of $H_1(\partial T, \mathbb{Z})$ and let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrix of ϕ . By assumption c is even. Since the matrix is in $SL(2, \mathbb{Z})$, a and d are both odd. Multiplying by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

if necessary, we may assume that b is also even. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}.$$

In this case the Mayer–Vietoris sequences of $M = T \cup N$ and $M_\phi = T \cup_\phi N$ are isomorphic with \mathbb{Z}_2 coefficients. This proves the second claim. \square

Unfortunately, the final conclusion of (2.4) does not characterize surgeries which are trivial mod 2. For instance, a surgery with coefficient $r = a/c$ on the unknot in S^3 results in a lens space S^3/\mathbb{Z}_a , cf. [Rolfsen76, 9.G.1]. Thus it does not change $H_1(M, \mathbb{Z}_2)$ iff a is odd; the parity of c does not matter.

Nonetheless, from the point of view of real algebraic geometry, (2.3) is the natural notion, cf. [Benedetti-Marin92, E.7].

One may ask which 3-manifolds can be obtained from a given one by surgeries which are trivial mod 2. [Benedetti-Marin92, Thm.C] gives a complete classification. The answer in the general case is somewhat complicated, but the orientable case is very easy to state:

Theorem 2.5. [Benedetti-Marin92] *Let M_1, M_2 be two compact orientable 3-manifolds. Then M_2 can be obtained from M_1 by a sequence of surgeries which are trivial mod 2 iff $h_1(M_1, \mathbb{Z}_2) = h_1(M_2, \mathbb{Z}_2)$.*

In the general case, we use the following consequence of the classification:

Theorem 2.6. [Benedetti-Marin92, Thm.B and Ex.C.5] *Any compact 3-manifold M can be obtained by a sequence of surgeries which are trivial mod 2 from a 3-manifold N such that either*

1. *N can be obtained from S^3 by repeatedly blowing up points and simple closed curves, or*
2. *N is a torus bundle over S^1 with monodromy $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in GL(2, \mathbb{Z})$.*

3. FLOPS

In this section we consider flops of real algebraic spaces. For the main theorem we need only the simplest type of flop and so we restrict our attention to flops of $(-1, -1)$ -curves. The general theory of flops on complex 3-folds is discussed in detail in [Kollár-Mori98].

3.1 (Contraction of ruled surfaces).

A *ruled surface* is a smooth, compact, complex surface F which can be written as a \mathbb{CP}^1 -bundle over a curve. We write it as $p : F \rightarrow C$ and a typical fiber is denoted by E .

F_m denotes the ruled surface obtained as the projectivization of the vector bundle $\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(m)$.

Let X' be a complex 3-fold and $F \subset X'$ a ruled surface. Let $N_{F|X'}$ denote the normal bundle of F . Assume that $(E \cdot N_{F|X'}) = -1$. By a theorem of [Nakano70] this implies that F can be contracted along E . That is, there is a complex 3-manifold Y containing a curve $C \subset Y$ and a proper morphism $f : X' \rightarrow Y$ such that

1. f restricts to an isomorphism $X' \setminus F \cong Y \setminus C$, and
2. $f|_F$ is the projection $p : F \rightarrow C$.

If X' is proper then so is Y . It is important to emphasize that even if X' is projective, Y is usually not projective. The projectivity of Y is a quite subtle question, see [Kollár91, Sec.4.2] for some results.

3.2 (Flops of $(-1, -1)$ -curves, complex case).

Let X be a complex 3-manifold and $\mathbb{CP}^1 \cong C \subset X$ a curve. Assume that the normal bundle of C in X is isomorphic to $\mathcal{O}_C(-1) + \mathcal{O}_C(-1)$. Let us blow up C in X . We obtain $p_1 : X_1 \rightarrow X$ with exceptional divisor $Q \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. Moreover, the normal bundle N_{Q, X_1} has intersection number -1 with both rulings of Q .

Thus $Q \subset X_1$ is symmetrical with respect to the two factors of Q and by (3.1) it can be contracted in the other direction to obtain $p_2 : X_1 \rightarrow X'$. The bimeromorphic map $\phi := p_2 \circ p_1^{-1} : X \dashrightarrow X'$ is called the *flop of C* . Frequently X' is also called a flop of X . Note that X and X' have a completely symmetrical role in a flop.

Definition 3.3. There are two different interval bundles over S^1 . The trivial one is obtained from $[0, 2\pi] \times [-1, 1]$ by identifying the points $(0, y) \leftrightarrow (2\pi, y)$. The total space is a *cylinder*. The nontrivial one is obtained from $[0, 2\pi] \times [-1, 1]$ by identifying the points $(0, y) \leftrightarrow (2\pi, -y)$. The total space is a *Möbius band*, it is not orientable. Its interior is called the open Möbius band. The image of the curve $[0, 2\pi] \times \{0\}$ is called the *center* of the cylinder or Möbius band. The cylinder and the Möbius band both retract to their center.

Let L be a line bundle of degree d on \mathbb{P}^1 defined over \mathbb{R} . (With a slight abuse of notation I will write $L \cong \mathcal{O}_{\mathbb{P}^1}(d)$.) Then $L(\mathbb{R})$ is an \mathbb{R}^1 -bundle over $\mathbb{RP}^1 \sim S^1$. It is the trivial bundle if d is even and the open Möbius band if d is odd.

3.4 (Flops of $(-1, -1)$ -curves, real case).

Notation as in (3.2). Assume in addition that X has a real structure and C is also real and isomorphic to \mathbb{P}^1 over \mathbb{R} , thus $C(\mathbb{R}) \sim S^1$. The isomorphism $N_{C|X} \cong \mathcal{O}_C(-1) + \mathcal{O}_C(-1)$ gives that

$$N_{C(\mathbb{R})|X(\mathbb{R})} \cong \mathcal{O}_{C(\mathbb{R})}(-1) + \mathcal{O}_{C(\mathbb{R})}(-1) \cong B_1 + B_2$$

where each B_i is an open Möbius band.

We can identify $N_{C(\mathbb{R})|X(\mathbb{R})}$ with a tubular neighborhood $T \supset C(\mathbb{R})$ such that ∂B_1 is a simple closed curve $J \subset \partial T$.

$X_1(\mathbb{R})$ is obtained from $X(\mathbb{R})$ by blowing up $C(\mathbb{R})$ which is contained in B_1 . Thus the birational transform $(p_1)_*^{-1} B_1$ is isomorphic to B_1 and it intersects $Q(\mathbb{R})$ in a fiber of p_2 . Hence the birational transform of B_1 in X' is a disc which intersects $C'(\mathbb{R})$ transversally. Thus $X(\mathbb{R}) \mapsto X'(\mathbb{R})$ is a surgery along $C(\mathbb{R})$ where J becomes the new meridian.

Definition–Lemma 3.5. A surgery $M \mapsto M_\phi$ along a knot $L \subset T \subset M$ is called a *topological flop* if the following equivalent conditions are satisfied:

1. $r(\phi) \equiv 1/2 \pmod{1}$.
2. There is an embedded Möbius band $B \subset T$ whose center is L such that ϕ maps the meridian of T to ∂B .

Proof. Let $J \subset \partial T$ be the image of the meridian under the surgery ϕ . If J is the boundary of an embedded Möbius band then $[J] = \pm 2[L]$ in $\pi_1(T)$, thus $c(\phi) = \pm 2$ and so $r(\phi) \equiv 1/2 \pmod{1}$. Conversely, if $J \subset \partial T$ is a simple closed curve in a homotopy class $(a, \pm 2)$ as in (2.1) then it is the boundary of the embedded Möbius band

$$[0, 2\pi] \times [-1, 1] \rightarrow D^2 \times S^1 \quad \text{given by} \quad (x, y) \mapsto (ye^{iax/2}, e^{ix}). \quad \square$$

Since a surgery is determined by the image of the meridian under ϕ , we obtain the following:

Proposition 3.6. *Let $L \subset M$ be a simple closed curve such that M is orientable along L . Then there is a one-to-one correspondence between*

1. *embedded Möbius bands $B \subset M$ with center L modulo isotopy, and*
2. *topological flops of L .* \square

Definition 3.7. Let M be a 3-manifold and $B \subset M$ an embedded Möbius band with center L . Assume that M is orientable along L . By (3.6) B defines a unique surgery $M \mapsto M'$. We call M' the *topological flop* of M along B .

The diffeomorphism class of M' is determined by the pair (M, B) up to isotopy.

The relationship between algebraic and topological flops is summarized in the next result which is a direct consequence of our previous discussions.

Proposition 3.8. *Let X be a 3-dimensional, smooth, real, algebraic space and $\mathbb{P}^1 \cong C \subset X$ a real curve. Assume that the normal bundle of C in X is isomorphic to $\mathcal{O}_C(-1) + \mathcal{O}_C(-1)$. Let $B \subset X(\mathbb{R})$ be a Möbius band corresponding to one of the $\mathcal{O}_C(-1)$ summands and $X \dashrightarrow X'$ the flop of C .*

Then $X'(\mathbb{R})$ is diffeomorphic to the topological flop of $X(\mathbb{R})$ along B . \square

The following proposition shows that the study of topological flops is equivalent to the study of surgeries which are trivial mod 2.

Proposition 3.9. *A surgery $M \mapsto M'$ which is trivial mod 2 can be written as a composite of topological flops.*

Proof. Let $M \mapsto M'$ correspond to the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } c \equiv 0 \pmod{2}.$$

By (6.3), we can write A as the product of matrices of the form

$$\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The first type of these corresponds to a change of the diffeomorphism $T \sim D^2 \times S^1$ and the second of these corresponds to a topological flop by (3.5). \square

4. ALGEBRAIC APPROXIMATION OF MÖBIUS BANDS

Theorem 4.1. *Let X be a compact, smooth, real, algebraic space which is birational to \mathbb{P}^3 . Let $B \subset X(\mathbb{R})$ be an embedded Möbius band. Then there is*

1. *a smooth, real, rational curve $\mathbb{P}^1 \cong C \subset X$ such that $C \cdot K_X \leq 0$, and*
2. *a real subline bundle $\mathcal{O}_C(-1) \cong M \subset N_{C|X}$*

such that the Möbius band corresponding to M is isotopic to B in $X(\mathbb{R})$.

Proof. Let $L \subset B$ denote the center of B . Choose a tubular neighborhood $L \subset T \subset X(\mathbb{R})$ such that $B \subset T$ and $\partial B \subset \partial T$ is a simple closed loop. Let $J \in H_1(\partial T, \mathbb{Z})$ denote its class.

If $L' \subset T$ is another knot which is isotopic to L by an isotopy inside T then T is also a tubular neighborhood of L' . Assume that $L' = C(\mathbb{R})$ for some algebraic curve $C \cong \mathbb{P}^1$ and $\mathcal{O}_C(-1) \cong M \subset N_{C|X}$ is a real subline bundle. M gives a Möbius band in T and hence a simple closed loop $J(M) \in H_1(\partial T, \mathbb{Z})$. Our aim is to find C and M such that $J(M) = J$. This will be done in several steps.

By [Bochnak-Kucharz99] there is a morphism $f : \mathbb{P}^1 \rightarrow X$ such that L is isotopic to $f(\mathbb{RP}^1)$ inside T . Moreover, we can assume that f^*T_X is ample. Indeed, by [Bochnak-Kucharz99, 2.5] we can assume that $f(\mathbb{RP}^1)$ passes through any number of preassigned points of $\text{Int } T$. Thus f^*T_X is ample by [Kollár96, II.3.10.1] for general choice of f .

Similarly, one can choose $C \subset X$ such that the normal bundle $N_{C|X}$ is the sum of two line bundles of large degree. This means that there are lots of injections $\mathcal{O}_C(-1) \hookrightarrow N_{C|X}$ and we can represent many Möbius bands. Unfortunately, one also needs to find a way to understand how the direct summands of $N_{C|X}$ twist compared to a fixed longitude of

L . I was unable to find a sensible way to do it, thus we proceed in a somewhat roundabout way.

4.2. Let E be a rank 2 complex vector bundle on \mathbb{P}^1 defined over \mathbb{R} . It induces a rank 2 real vector bundle $E(\mathbb{R}) \rightarrow \mathbb{RP}^1$. Let $L \subset E(\mathbb{R})$ be the zero section and $L \subset T \subset E(\mathbb{R})$ a tubular neighborhood. We saw in (2.1) that there is a canonical choice for a meridian $m \in H_1(\partial T, \mathbb{Z})$ (up to sign). The real algebraic structure makes it possible to choose a longitude as well in some cases.

Assume that $E(\mathbb{R})$ is orientable. We can write $E \cong \mathcal{O}_{\mathbb{P}^1}(a) + \mathcal{O}_{\mathbb{P}^1}(b)$ for some a, b where $a \equiv b \pmod{2}$ by the orientability assumption.

Assume first that a, b are both even and $a \leq b$. Then $(0, (s^2 + t^2)^{b/2})$ is a nowhere zero section of $E(\mathbb{R})$, thus it defines a longitude $\ell \in H_1(\partial T, \mathbb{Z})$ (up to sign). $\{m, \ell\}$ is a basis of $H_1(\partial T, \mathbb{Z})$.

The situation is a bit more complicated if $a \leq b$ are both odd. In this case set

$$H_1(\partial T, \mathbb{Z})^{(2)} := \{c \in H_1(\partial T, \mathbb{Z}) : (m \cdot c) \equiv 0 \pmod{2}\}.$$

$H_1(\partial T, \mathbb{Z})^{(2)}$ is an index 2 subgroup of $H_1(\partial T, \mathbb{Z})$. If $B \subset T$ is a Möbius band with center L then $[\partial B] \in H_1(\partial T, \mathbb{Z})^{(2)}$. The $\mathcal{O}_{\mathbb{P}^1}(b)$ summand corresponds to a such a Möbius band. This gives a well defined element $\ell^{(2)} \in H_1(\partial T, \mathbb{Z})^{(2)}$ (up to sign) and $\{m, \ell^{(2)}\}$ is a basis of $H_1(\partial T, \mathbb{Z})^{(2)}$. In this case set $\ell := \frac{1}{2}\ell^{(2)} \in H_1(\partial T, \mathbb{Q})$.

Lemma 4.3. *Notation as above. A given $d \in H_1(\partial T, \mathbb{Z})$ can be represented as the boundary of a Möbius band corresponding to an embedding $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow E$ if d is not divisible by 2, $m \cdot d = \pm 2$ and $|\ell \cdot d| \leq \min\{a, b\}$.*

Proof. Assume first that a, b are both even. A map

$$j : \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{(f,g)} \mathcal{O}_{\mathbb{P}^1}(a) + \mathcal{O}_{\mathbb{P}^1}(b)$$

is given by a pair of homogeneous polynomials of degrees $a+1, b+1$. The image of j is a subline bundle iff f and g have no common zeros. Let $J(j) \subset \partial T$ be the curve corresponding to $\text{im } j$.

The intersection points of $J(j)$ and ℓ correspond to the zeros of f . Choose f and g such that they both have $c \leq 1 + \min\{a, b\}$ real roots and these roots alternate in \mathbb{RP}^1 . It is then easy to see that $|\ell \cdot J(j)| = c$. Replacing g by $-g$ will change the sign.

The case when a, b are both odd are similar. \square

Let $f : \mathbb{P}^1 \rightarrow X$ be a morphism which is an embedding in a complex analytic neighborhood of \mathbb{RP}^1 . Then $f(\mathbb{RP}^1) \subset X(\mathbb{R})$ is a knot and the notions of tubular neighborhood, meridian and longitude make sense

as before. Instead of a normal bundle we have a normal sheaf $N_f := f^*T_X/T_{\mathbb{P}^1}$ which can be written as

$$N_f = \mathcal{O}_{\mathbb{P}^1}(a_f) + \mathcal{O}_{\mathbb{P}^1}(b_f) + (\text{torsion}),$$

where the support of the torsion is disjoint from $\mathbb{R}\mathbb{P}^1$. Let $C \subset X$ be a smooth rational curve such that $N_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(a) + \mathcal{O}_{\mathbb{P}^1}(b)$. Consider the composite

$$f_m : \mathbb{P}^1 \xrightarrow{\phi_m} \mathbb{P}^1 \cong C$$

where ϕ_m is as in (4.4). Then

$$N_{f_m} = \mathcal{O}_{\mathbb{P}^1}(ma) + \mathcal{O}_{\mathbb{P}^1}(mb) + (\text{torsion}).$$

Assume that $a, b > 0$ and let $B \subset X(\mathbb{R})$ be a Möbius band with center $C(\mathbb{R})$. Then for $m \gg 1$ there is a subline bundle $j_m : \mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow N_{f_m}$ such that the Möbius band corresponding to j_m is isotopic to B .

This is essentially what we want, except that f_m is not an embedding. We claim, however, that a small perturbation of f_m gives a solution of (4.1).

By [Kollár96, II.3.14.3] there is a morphism $F : \Delta \times \mathbb{P}^1 \rightarrow X$ such that $F|_{\{0\} \times \mathbb{P}^1} = f_m$ and $F|_{\{t\} \times \mathbb{P}^1}$ is an embedding for $t \neq 0$.

The normal sheaf $N_F := F^*T_X/T_{\Delta \times \mathbb{P}^1}$ is flat over Δ and so the injection

$$j : \mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow N_{f_m} \cong N_F|_{\{0\} \times \mathbb{P}^1}$$

extends to injections

$$j_t : \mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow N_F|_{\{t\} \times \mathbb{P}^1} \cong N_{C_t|X}$$

since $H^1(\mathbb{P}^1, N_{f_m}(1)) = 0$. □

Lemma 4.4. $x \mapsto x + (x^2 + 1)^{-m}$ extends to a morphism $\phi_m : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $2m + 1$. $\phi_m : \mathbb{R}\mathbb{P}^1 \rightarrow \mathbb{R}\mathbb{P}^1$ is a homeomorphism and ϕ_m is unramified along $\mathbb{R}\mathbb{P}^1$. □

5. ALGEBRAIC REALIZATION OF TOPLOGICAL FLOPS

Theorem 5.1. *Let X be a compact, smooth, real, algebraic space and $\mathbb{P}^1 \cong C \subset X$ a smooth, real, rational curve such that $C \cdot K_X \leq 0$. Let $\mathcal{O}_C(-1) \subset N_{C|X}$ be a real subline bundle and $C(\mathbb{R}) \subset B \subset X(\mathbb{R})$ the corresponding Möbius band. Assume that $X(\mathbb{R})$ is orientable along $C(\mathbb{R})$. Then there is*

1. *a blow up $p : X_1 \rightarrow X$ such that $X_1(\mathbb{R}) \rightarrow X(\mathbb{R})$ is a homeomorphism, and*
2. *a flop $\pi : X_1 \dashrightarrow X_2$ of $p_*^{-1}(C)$*

such that

3. X_2 is a compact, smooth, real, algebraic space birational to X ,
and
4. $X_2(\mathbb{R})$ is diffeomorphic to the topological flop of $X(\mathbb{R})$ along B .

Proof. $X(\mathbb{R})$ is orientable along $C(\mathbb{R})$, thus $w_1(X(\mathbb{R})) \cdot C(\mathbb{R}) = 0$ where w_1 denotes the first Stiefel–Whitney class (cf. [Milnor-Stasheff74]).

$$c_1(X(\mathbb{C})) \cdot C \equiv w_1(X(\mathbb{R})) \cdot C(\mathbb{R}) \pmod{2}$$

hence

$$\deg N_{C|X} = -K_X \cdot C - 2 = c_1(X(\mathbb{C})) \cdot C - 2 \equiv 0 \pmod{2}.$$

Thus $L' := N_{C|X}/L$ is a line bundle of degree $-1 + 2r$ for some $r \geq 0$. By (5.3) there is a smooth, real, algebraic curve $D \subset X$ such that

1. $D(\mathbb{R}) = \emptyset$,
2. $D \cap C$ is precisely r conjugate point pairs $P_1, \bar{P}_1, \dots, P_r, \bar{P}_r$, and
3. the tangent vector of D at every point P_i, \bar{P}_i maps to a nonzero vector in $L \subset N_{C|X}$.

Let us blow up D to obtain $p : X_1 := B_D X \rightarrow X$ and set $C_1 := p_*^{-1}C$. There is an exact sequence

$$0 \rightarrow L \rightarrow N_{C_1|X_1} \rightarrow L'(-\sum(P_i + \bar{P}_i)) \rightarrow 0$$

Since $\deg L'(-\sum(P_i + \bar{P}_i)) = -1 + 2r - 2r = -1$, the sequence splits and

$$N_{C_1|X_1} \cong \mathcal{O}_{C_1}(-1) + \mathcal{O}_{C_1}(-1).$$

Thus $C_1 \subset X_1$ can be flopped to get X_2 . (5.1.4) follows from (3.8). \square

5.2 (General position curves in algebraic spaces).

If X is a quasi projective variety, it is easy to construct a curve D as above by taking the intersection of suitable hypersurfaces. On an algebraic space there may not exist any base point free linear system, so intersections of hypersurfaces are unlikely to produce the needed curve D . In fact, there are singular algebraic spaces with a unique singular point $P \in X$ such that every curve in X passes through P . No such smooth examples are known and it is conjectured that this can not happen for smooth algebraic spaces. (See [Kollár91, 5.2.6] for the case when $\dim X \leq 3$.)

The next lemma shows the existence of the required curve D in some special cases which are sufficient for our purposes. Note also that the nonprojective case is needed only for 3-manifolds as in (2.6.2). In section 6 we give an explicit construction for such a real algebraic space. It is not hard to use this construction to show the existence of the curve D .

Lemma 5.3. *Let X be a smooth, proper, real algebraic space of dimension $n \geq 3$ which is birational to \mathbb{P}^n . Let $Z \subset X$ be a subset of codimension at least 2, $p_i, \bar{p}_i \in X \setminus Z$ conjugate pairs of points and $0 \neq v_i \in T_{p_i}X$ tangent vectors. Then there is a smooth, proper, real algebraic curve $D \subset X$ such that*

1. $D(\mathbb{R}) = \emptyset$,
2. $D \cap Z = \emptyset$,
3. $p_i \in D$ and $T_{p_i}D = \mathbb{C} \cdot v_i$ for every i .

Proof. First we find morphisms, defined over \mathbb{C} , $g_i : \mathbb{CP}^1 \rightarrow X(\mathbb{C})$ whose image passes through p_i and has tangent space $\mathbb{C} \cdot v_i$ there. Then we prove that if we choose the g_i sufficiently general then the union of $\text{im } g_i$ and their conjugates is a curve D as desired. A modification of this construction would give an irreducible D as well, but this is not important for us.

Let $p = p_i$ be any of the points. By [Kollár96, II.1.5] there is an algebraic space U parametrizing all morphisms $\mathbb{CP}^1 \rightarrow X$ which pass through p and have tangent space $\mathbb{C} \cdot v$ at p . Let $F : U \times \mathbb{P}^1 \rightarrow X$ be the universal morphism. For $u \in U$ set $F_u := F|_{\{u\} \times \mathbb{P}^1}$.

By [Kollár96, IV.3.9] there is a $u \in U$ such that $F_u^*T_X$ is ample. (Strictly speaking, we need to apply the refined version [Kollár98a, 4.1.2].) By composing F_u with a multiple cover $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ we may assume that even $F_u^*T_X(-2)$ is ample. Hence there is a connected open subset $W \subset U$ such that $F_w^*T_X(-2)$ is ample for every $w \in W$.

As in [Kollár96, II.3.7] this implies the following general position statements for W :

1. For any $x \in X \setminus \{p\}$ the set $W_x := \{w \in W | x \in \text{im } F_w\}$ has complex codimension $n - 1$ in W .
2. For any $x_1 \neq x_2 \in X \setminus \{p\}$ the set $W_{x_1, x_2} := \{w \in W | x_1, x_2 \in \text{im } F_w\}$ has complex codimension $2n - 2$ in W .

I claim that $\text{im } F_w$ satisfies (5.3.1–2) for general $w \in W$. Indeed, the set of all maps intersecting Z is $\cup_{x \in Z} W_x$ and this has codimension $\geq (n - 1) - (n - 2) = 1$. The set of those maps which intersect $X(\mathbb{R})$ has real codimension $\geq (2n - 2) - n \geq 1$.

Moreover, the set of those maps which pass through a conjugate point pair has codimension $\geq (2n - 2) - n \geq 1$, hence $\text{im}(F_w)$ and $\overline{\text{im}(F_w)}$ are disjoint for general $w \in W$.

To complete the proof, first let $p = p_1$ and set $D_1 := \text{im}(F_w) \cup \overline{\text{im}(F_w)}$ for suitable general $w \in W$.

Next let $p = p_2$ and set $Z_1 = Z \cup D_1$. As before we obtain D_2 and we can proceed to find the rest of the D_i inductively. Finally set $D = \cup_i D_i$. \square

6. TORUS BUNDLES

The aim of this section is to exhibit examples of real algebraic spaces X such that X is birational to \mathbb{P}^3 and $X(\mathbb{R})$ is a torus bundle over a circle with prescribed monodromy. We start with some easy results about subgroups of $GL(2, \mathbb{Z})$.

6.1 (Subgroups of $GL(2, \mathbb{Z})$).

Let $H_1 \subset GL(2, \mathbb{Z})$ denote the subgroup of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } c \equiv 0 \pmod{2}.$$

Let $H_2 \subset GL(2, \mathbb{Z})$ denote the subgroup of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}.$$

The matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

generate a 6 element subgroup $K_6 \subset GL(2, \mathbb{Z})$ and $GL(2, \mathbb{Z})$ is the semidirect product of H_2 by K_6 .

Let $H_4 \subset GL(2, \mathbb{Z})$ denote the subgroup of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } a, d \equiv 1 \pmod{4} \text{ and } b, c \equiv 0 \pmod{2}.$$

The matrices $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ form a 4 element subgroup $K_4 \subset H_2$ and H_2 is the semidirect product of H_4 by K_4 .

Lemma 6.2. H_4 is generated by the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Proof. Right multiplication by these matrices and their inverses changes the first row (a, b) of a matrix to $(a, b \pm 2a)$ or to $(a \pm 2b, b)$. This allows us to run a version of the Euclidean algorithm on the pair (a, b) . Taking into account that a and b are relatively prime, $a \equiv 1 \pmod{4}$ and b is even, we see that we can always reduce to the case when the first row is $(1, 0)$. Such a matrix in H_4 is automatically a power of the second matrix above. \square

This easily implies the following:

Corollary 6.3. H_1 is generated by $\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. \square

6.4 (Birational transformations along ruled surfaces).

Let X be a complex 3-fold and $F \subset X$ a ruled surface with normal bundle $N_{F|X}$. Assume that $(E \cdot N_{F|X}) = 0$ where $E \subset F$ is a fiber of the projection $p : F \rightarrow C$. Let $D \subset F$ be a section of p ; thus $D \cong C$ and $(D \cdot E) = 1$. Let $X' = B_D X$ be the blow up of D with projection $g : X' \rightarrow X$. We blow up a curve inside F , so the birational transform $g_*^{-1}(F)$ of F is isomorphic to F . If N denotes the normal bundle of $g_*^{-1}(F) \subset X'$ then $N \cong N_{F|X}(-D)$. Thus $(E \cdot N) = -1$. By (3.1) $g_*^{-1}(F)$ can be contracted and we obtain a birational transformation $\rho : X \dashrightarrow Y$. ρ is the composite of a blow up of the curve D followed by the inverse of the blow up of a curve C . As in (3.1) Y need not be projective, even if X is.

6.5 (Birational transformations of quadric bundles, local case).

The smooth quadric surface $Q \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ is ruled in two distinct ways. Let $p : Q \rightarrow \mathbb{CP}^1$ be the second projection and $E = \mathbb{CP}^1 \times (0 : 1)$ a fiber. Set $F = (0 : 1) \times \mathbb{CP}^1$ and let $D \subset Q$ denote a section of p . D is linearly equivalent to $mE + F$ for some $m \geq 0$.

Let $0 \in \Delta$ be the complex unit disc and consider the complex 3-manifold $X := Q \times \Delta$ with second projection $\pi : X \rightarrow \Delta$. Set $F = Q \times \{0\}$. Then $D \subset F \subset X$ satisfy the assumptions of (6.4), hence we can construct a birational transformation $\rho : X \dashrightarrow Y$.

$(D \cdot D) = 2m$ and so the normal bundle of D in X is $\mathcal{O}_D + \mathcal{O}_D(2m)$. Hence the exceptional divisor of the blow up $g : X' \rightarrow X$ is isomorphic to the ruled surface F_{2m} . We can now contract the birational transform $g_*^{-1}F$ to obtain Y . All these transformations take place over the central fiber of π , so we still have a smooth, proper morphism $\Pi : Y \rightarrow \Delta$. We have not changed the family over $\Delta \setminus \{0\}$ but the fiber of Π over 0 is now F_{2m} .

Assume now that everything is defined over \mathbb{R} . $X(\mathbb{R}) \rightarrow (-1, 1)$ is a trivial torus bundle. $Y(\mathbb{R}) \rightarrow (-1, 1)$ is still smooth and proper, so it is again a trivial torus bundle. We have, however, changed the trivialization.

In order to compute the change in the trivialization, let $b^+, b^- \in (-1, 1)$ be a positive and a negative point. Fix bases in $H_1(\pi^{-1}(b^\pm), \mathbb{Z})$ by setting

$$e^\pm = E(\mathbb{R}) \times (0 : 1) \times \{b^\pm\} \quad \text{and} \quad f^\pm = (0 : 1) \times F(\mathbb{R}) \times \{b^\pm\};$$

in both cases using the positive orientation on $E(\mathbb{R})$ and $F(\mathbb{R})$. As we go from b^- to b^+ , parallel transport produces an isomorphism

$$T_X : H_1(\pi^{-1}(b^-), \mathbb{Z}) \rightarrow H_1(\pi^{-1}(b^+), \mathbb{Z})$$

whose matrix is the identity.

$Y(\mathbb{R}) \rightarrow (-1, 1)$ is another torus bundle, so we get another parallel transport isomorphism

$$T_Y : H_1(\Pi^{-1}(b^-), \mathbb{Z}) \rightarrow H_1(\Pi^{-1}(b^+), \mathbb{Z}).$$

Let us compute the matrix of T_Y .

The homology class of $D(\mathbb{R}) \times \{b^\pm\}$ is of the form $ne^\pm + f^\pm$ for some $|n| \leq m$. It is easy to see that any n can occur for suitable choice of D . For our purposes we need only the cases $n \in \{-1, 0, 1\}$. These are realized by the skew diagonal, F and the diagonal.

$D \times (-1, 1)$ intersects F transversally in $D \times \{0\}$, so $g_*^{-1}(D \times (-1, 1)) \cong D \times (-1, 1)$ is disjoint from $g_*^{-1}(Q)$. Thus T_Y transports $D(\mathbb{R}) \times \{b^-\}$ to $D(\mathbb{R}) \times \{b^+\}$ and preserves the orientation, so

$$T_Y \begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ 1 \end{pmatrix}.$$

Consider now $E \times (-1, 1)$. We blow up the unique point $(E \cap D) \times \{0\}$ and then contract the birational transform of $E \times \{0\}$. Let t be a local coordinate on Δ and $(x_0 : x_1)$ coordinates on E such that $E \cap D = (0 : 1)$. Then the curves $x_0 = \lambda x_1$ give the parallel transport before blow up and the curves $x_0 = \lambda t x_1$ give the parallel transport after blow up. Thus T_Y transports $E(\mathbb{R}) \times \{b^-\}$ to $E(\mathbb{R}) \times \{b^+\}$ but it reverses the orientation. Therefore

$$T_Y \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and so} \quad T_Y = \begin{pmatrix} -1 & 2n \\ 0 & 1 \end{pmatrix}.$$

Of course we could have used the other projection of the quadric as a ruled surface structure. These give parallel transport matrices

$$T'_Y = \begin{pmatrix} 1 & 0 \\ 2n & -1 \end{pmatrix}.$$

6.6 (Birational transformations of quadric bundles, global case).

Let $X = Q \times \mathbb{CP}^1$ with projection $\pi : X \rightarrow \mathbb{CP}^1$. We can pick points $p_i \in \mathbb{RP}^1$ and at each of these points perform one of the birational transformations discussed above. At the end we obtain a real algebraic space Y such that the projection $\Pi : Y(\mathbb{R}) \rightarrow \mathbb{RP}^1$ is a torus bundle.

The monodromy action on H_1 can be written as the product of the local parallel transport matrices T_Y and T'_Y . Observe that

$$\begin{pmatrix} -1 & \pm 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}.$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}.$$

Thus by (6.1) and (6.2) the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}$$

generate $H_2 \subset GL(2, \mathbb{Z})$. Hence we get examples of real algebraic spaces Y such that Y is birational to \mathbb{P}^3 and $Y(\mathbb{R})$ is a torus bundle over a circle with arbitrary monodromy in H_2 .

Unfortunately, the example we need in (2.6.2) can not be realized with monodromy in H_2 . Thus we need to introduce a further twist in the construction.

6.7 (An example of a degree 6 Del Pezzo surface).

Let Q be the quadric surface $(x^2 + y^2 + z^2 = t^2) \subset \mathbb{P}^3$ and $p_1, p_2, p_3 \in Q(\mathbb{R})$ real points. Let $C \subset Q$ denote the intersection of Q with the plane spanned by the points p_1, p_2, p_3 . Let us blow up the 3 points to obtain $h : T \rightarrow Q$ with exceptional curves E_i . $h_*^{-1}(C)$ is a smooth rational curve with selfintersection -1 , thus it can be contracted $g : T \rightarrow S$. S is a degree 6 Del Pezzo surface. Set $F_i := g_*(E_i)$.

$T(\mathbb{R})$ is the connected sum of 3 copies of \mathbb{RP}^2 . The birational transform of $C(\mathbb{R})$ intersects each $E_i(\mathbb{R})$ in a single point, thus $S(\mathbb{R})$ is a torus.

Fix an orientation on $Q(\mathbb{R}) \sim S^2$. This induces an orientation on each $E_i(\mathbb{R})$. $C(\mathbb{R})$ is a circle which divides $Q(\mathbb{R})$ into two discs. The image of either of these discs in $S(\mathbb{R})$ has $\cup F_i(\mathbb{R})$ as its boundary. Thus $\sum [F_i(\mathbb{R})] = 0 \in H_1(S(\mathbb{R}), \mathbb{Z})$. The intersection number of $F_i(\mathbb{R})$ and of $F_j(\mathbb{R})$ is ± 1 , hence any two of the $[F_i(\mathbb{R})]$ form a basis of $H_1(S(\mathbb{R}), \mathbb{Z})$.

Assume now that we have a diffeomorphism $Q(\mathbb{R}) \rightarrow Q(\mathbb{R})$ which preserves orientation and permutes the three points $(p_1, p_2, p_3) \mapsto (p_2, p_3, p_1)$. Using $[F_1(\mathbb{R})], -[F_2(\mathbb{R})]$ as a basis, the induced map on $H_1(S(\mathbb{R}), \mathbb{Z})$ is given by the matrix

$$\tau_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Lemma 6.8. *Let $K_6 \subset GL(2, \mathbb{Z})$ be the subgroup defined in (6.1). For every $M \in K_6$ there is a smooth, projective, real algebraic variety X such that*

1. *there is a morphism $X \rightarrow \mathbb{P}^1$ which is an analytically locally trivial fiber bundle over \mathbb{RP}^1 ,*
2. *$X(\mathbb{R}) \rightarrow \mathbb{RP}^1$ is a torus bundle with monodromy M , and*
3. *X is obtained from $\mathbb{P}^2 \times \mathbb{P}^1$ by a sequence of smooth blows ups and blow downs.*

Proof. First we give an example with order 3 monodromy in K_6 . Fix a smooth conic $C \subset Q$ and consider the surface $C \times \mathbb{P}^1$. Let $D \subset C \times \mathbb{P}^1$ be the graph of the morphism $C \cong \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given as $z \mapsto (z^3 - 4z)/(z^2 - 1)$. The induced projection $D \rightarrow \mathbb{P}^1$ has degree 3 and $D(\mathbb{R}) \rightarrow \mathbb{RP}^1$ is three-to-one over every point. Let $Y \rightarrow Q \times \mathbb{P}^1$ be the blow up of D . The birational transform of $C \times \mathbb{P}^1$ is contractible in Y . By contracting it we obtain $Y \rightarrow X \rightarrow \mathbb{P}^1$.

Over each real point we have an example of the construction of a degree 6 Del Pezzo surface given in (6.7), thus $X(\mathbb{R}) \rightarrow \mathbb{RP}^1$ is a torus bundle with order 3 monodromy τ_3 as above. The quadric Q is obtained from \mathbb{P}^2 by two blow ups and one blow down, so $Q \times \mathbb{P}^1$ is obtained from $\mathbb{P}^2 \times \mathbb{P}^1$ by two smooth blow ups and one smooth blow down.

Examples with order 2 monodromy can also be obtained from degree 6 Del Pezzo surfaces, but they are easier to get from a family of quadrics. We again start with $C \times \mathbb{P}^1$ but we realize C as a line in \mathbb{P}^2 . Choose $D \subset C \times \mathbb{P}^1$ as the graph of the morphism $C \cong \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given as $z \mapsto (z^2 - 1)/z$. The previous blow up and blowdown construction gives an example with monodromy

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have to perform one more transformation as in (6.5) with $n = 0$ to get an example with monodromy

$$\tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrices τ_3 and τ_2 , together with their conjugates, give all elements of K_6 . \square

6.9 (Birational transformations along degree 6 Del Pezzo surfaces).

Let X be a complex 3-fold with real structure and $F \subset X$ a surface isomorphic to S defined in (6.7). Assume that F has trivial normal bundle $N_{F|X} \cong \mathcal{O}_F$. Then $F(\mathbb{R})$ looks like a ruled surface. We would like to perform the same transformations we did in (6.4). In order to do this, we need to perform some preparatory transformations first. During these we do not change the real points but the surface F becomes a ruled surface with the requisite normal bundle. This is done in two steps.

Through each point $p_i \in Q$ there is a conjugate pair of lines. Let $G_i \subset F \cong S$ denote their birational transform.

First blow up $G_1 \subset F$ to get X_1 and let $F_1 \subset X_1$ be the birational transform of F . Then $F_1 \cong F$ and $G_2 \subset F_1 \subset X_1$ is a conjugate pair of smooth rational curves with normal bundles $\mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}(-1)$. Thus

we can flop $G_2 \subset X_1$ to obtain X_2 . Let $F_2 \subset X_2$ denote the birational transform of F_1 . F_2 is obtained from $F_1 \cong S$ by contracting G_2 , thus F_2 can be identified with $\mathbb{P}^1 \times \mathbb{P}^1$. The transformations $X \dashrightarrow X_1 \dashrightarrow X_2$ are isomorphisms along the real points. Hence we are in the situation of (6.4) and we can perform the transformations discussed there.

Putting all of these together, we obtain the main result of the section:

Theorem 6.10. *For every $M \in GL(2, \mathbb{Z})$ there is a smooth, real, algebraic space X such that*

1. $X(\mathbb{R})$ is a torus bundle over S^1 with monodromy M , and
2. X is obtained from $\mathbb{P}^2 \times \mathbb{P}^1$ by a sequence of smooth blow ups and blow downs.

Proof. Write $M = M_1 M_2$ where $M_1 \in K_6$ and $M_2 \in H_2$. We also write $M_2 = \prod T_i$ where each T_i is one of the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}.$$

By (6.8) we can obtain an X_1 as in (6.10) with monodromy M_1 . Pick points in cyclic order $p_i \in \mathbb{RP}^1$. $X_1 \rightarrow \mathbb{P}^1$ is a trivial S -bundle over an analytic neighborhood of p_i , thus we can perform birational transformations as in (6.9) to change the global monodromy by the matrix T_i . Doing this at all points p_i , we obtain $X \rightarrow \mathbb{P}^1$. The monodromy of $X(\mathbb{R}) \rightarrow \mathbb{RP}^1$ is thus $M_1 \prod T_i = M$. \square

Remark 6.11. From (6.10) we only need the special case when

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}.$$

In this case we need to perform the birational transformations in a single fiber only. It is not hard to see that during this process we create an effective curve which is homologous to zero, so the resulting algebraic space X is not projective.

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